Reynolds transport theorem

Reynolds transport theorem (also known as the Leibniz-Reynolds transport theorem), or in short **Reynolds theorem**, is a three-dimensional generalization of the Leibniz integral rule. This theorem is used to compute derivatives of integrated quantities.

Reynolds transport theorem can be simply stated as - What was already there plus what goes in minus what comes out is equal to what is there. Reynolds theorem is used in formulating the basic conservation laws of continuum mechanics, particularly fluid dynamics and large-deformation solid mechanics. These conservation laws (law of conservation of mass, law of conservation of linear momentum, and law of conservation of energy) are adopted from classical mechanics and thermodynamics where the system approach is normally followed. In fluid mechanics, it is often more convenient to work with control volumes as it is difficult to identify and follow a system of fluid particles. Thus, there is a need to relate the system equations and corresponding control volume equations. The link between the two is given by the Reynolds transport theorem. The theorem is named after Osborne Reynolds (1842–1912).

Imagine a system and a coinciding control volume with a control surface. Reynolds transport theorem states that the rate of change of an extensive property *N*, for the system is equal to the time rate of change of *N* within the control volume and the net rate of flux of the property *N* through the control surface. For an example, the law of conservation of mass states that rate of change of the property, mass, is equal to the sum of the rate of accumulation of mass within a control volume and the net rate of flow of mass across the control surface.

The differential forms of these equations with additional assumption of Newton's viscosity law are commonly known as the Navier-Stokes equations.

General form

The Reynolds transport theorem refers to any extensive property, *N*, of the fluid in a particular control volume. It is expressed in terms of a substantive derivative on the left-hand side.

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta)dV + \int_{c.s.} \rho\eta\vec{v}_b \cdot \hat{n}dA + \int_{c.s.} \rho\eta\vec{v}_r \cdot \hat{n}dA,$$

where η is the intensive property related to extensive property *N*, (*N* per unit mass), *t* is time, *c.v.* refers to the control volume, *c.s.* refers to the control surface, ρ is the fluid density, *V* is the volume, υ_b is the velocity of the boundary of the control volume (the control surface), υ_r is the velocity of the fluid with respect to the control surface, *n* is the outward pointing normal vector on the control surface, and *A* is the area.

Mass formulation

Also called the continuity equation, the control volume form of the conservation of mass is found by substituting mass in for *N*. This means that η is equal to 1.

$$0 = \int_{c.v.} \frac{\partial\rho}{\partial t}dV + \int_{c.s.} \rho\vec{v}_b \cdot \hat{n}\, dA + \int_{c.s.} \rho\vec{v}_r \cdot \hat{n}\, dA$$

All variables are defined as in the general formulation. *M* is equal to the mass of the control volume. Applying the Conservation of mass principle, the left hand side reduces to 0 since mass of a system cannot change in time. In a steady flow system, the first term on the right hand side of the equation will be equal to 0, i.e. the mass of the control volume does not change, implying that the mass flow rate into the control volume is equal to the mass flow rate out of the control volume.

Momentum formulation

The momentum equation is found by substituting momentum in for *N*. From this, η is found to be velocity. From Newton's second law, we have the time rate of change of momentum (now the left hand side of the equation) is equal to the net force. Thus,

$$\sum \vec{F} = \int_{c.v.} \frac{\partial}{\partial t}(\rho \vec{v}) dV + \int_{c.s.} \rho \vec{v}(\vec{v}_b \cdot \hat{n}) dA + \int_{c.s.} \rho \vec{v}(\vec{v}_r \cdot \hat{n}) dA,$$

where *F* is force, υ is the velocity of fluid in a coordinate system attached to the control surface, and all other variables are defined as in the general formulation. Note that the integral form of the momentum equation is a vector equation.

Energy formulation

The energy equation is found by substituting energy in for *N*. From this, η is found to be energy per unit mass.

$$\dot{Q} - \sum \dot{W} = \int_{c.v.} \frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + gz + \tilde{u}\right)\right] dV + \int_{c.s.}\left[\frac{v^2}{2} + gz + \tilde{u} + \frac{p}{\rho}\right]\rho \vec{v}_b \cdot \hat{n} dA + \int_{c.s.}\left[\frac{v^2}{2} +\right.$$

where *Q* is the heat transfer into the control volume, *W* is the work done by the system, *g* is the acceleration due to gravity, *z* is the vertical distance from an arbitrary datum, $\tilde{u}$ is the specific internal energy of the fluid, *p* is the pressure and all other variables are defined as in the general formulation.

Note that these equations make no consideration for chemical reactions or potential energy associated with electromagnetic fields.

Formulation used in solid mechanics

Suppose $\Omega(t)$ is a region in Euclidean space with boundary $\partial\Omega(t)$, and let $\mathbf{n}(\mathbf{x},t)$ be the outward unit normal to the boundary at time t. Let $\mathbf{x}(t)$ be the positions of points in the region, $\mathbf{v}(\mathbf{x},t)$ the velocity field in the region, and let $\mathbf{f}(\mathbf{x},t)$ be a vector field in the region (it may also be a scalar field). Reynolds transport theorem states that[1]

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}\ \mathrm{dV}\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\ \mathrm{dV} + \int_{\partial\Omega(t)} (\mathbf{v}\cdot\mathbf{n})\mathbf{f}\ \mathrm{dA}\ .$$